ISSN 1810-0198. Вестник Тамбовского университета. Серия Естественные и технические науки

Том 23, № 122 2018

DOI: 10.20310/1810-0198-2018-23-122-131-135

EXISTENCE AND STABILITY OF BUMPS IN A NEURAL FIELD MODEL

© K. Kolodina[1)], A. Oleynik[2)], J. Wyller[1)]

[1)] Norwegian University of Life Sciences
3 Universitetstunet, Ås 1433, Norway
E-mail: karina.kolodina@nmbu.no, john.wyller@nmbu.no
[2)] University of Bergen
41 Allegaten, Realfagbygget, Bergen 5020, Norway
E-mail: Anna.Oleynik@uib.no

Abstract. We investigate existence and stability of *bumps* (localized stationary solutions) in a homogenized 2-population neural field model, when the firing rate functions are given by the unit step function.

Keywords: homogenization theory; existence and stability of stationary solutions of nonlocal neural field models

1. The main results

The set of coupled integro-differential equations

$$\begin{aligned} \frac{\partial}{\partial t}u_e &= -u_e + \omega_{ee} \otimes \otimes P_e(u_e - \theta_e) - \omega_{ie} \otimes \otimes P_i(u_i - \theta_i) \\ \tau \frac{\partial}{\partial t}u_i &= -u_i + \omega_{ei} \otimes \otimes P_e(u_e - \theta_e) - \omega_{ii} \otimes \otimes P_i(u_i - \theta_i) \end{aligned} \tag{1}$$

models the neural activity in the cortical tissue. Here $f \otimes \otimes g$ is is defined as

$$[f \otimes \otimes g](x, y) \equiv \int\limits_{\Omega} \int\limits_{Y} f(x - x', y - y')g(x', y')dy'dx'$$

where $x \in \Omega \subseteq \Re$, $y \in \Re$, $t > 0$. u_e and u_i are the membrane potentials of excitatory and inhibitory neurons, respectively, at the spatial point x, y and time t. The region Ω is the spatial region occupied by the neurons. The functions ω_{mn} $(m, n = e, i)$ model the coupling strengths (referred to as the connectivity functions) in the network. The functions $P_m, m = e, i$ (referred to as the firing rate functions) are monotonically increasing and assume values in the interval $Y = [0, 1]$. The connectivity functions are assumed to be 1-periodic in the variable y. The parameter τ is the relative inhibition time i.e. $\tau = \tau_i/\tau_e$

The work is partially supported by the Norwegian University of Life Sciences and The Research Council of Norway (project № 239070).

where τ_e (τ_i) is the excitatory (inhibitory) time constant, while θ_e and θ_i are the threshold values for firing of the excitatory and the inhibitory neurons, respectively. The system (1) is derived from a 2-population neural field model with periodic microstructure incorporated in the connectivity functions by using the 2-scale convergence technique of Nguetseng [1].

The connectivity kernels ω_{mn} are expressed in terms of the scaling function Φ and the footprint functions σ_{mn}, $m,n=e,i$:

$$\begin{gathered}\omega_{mn}(x,y;\alpha_{mn}) = \tfrac{1}{\sigma_{mn}(y;\alpha_{mn})}\Phi\left(\tfrac{x}{\sigma_{mn}(y;\alpha_{mn})}\right)\\ \sigma_{mn}(y;\alpha_{mn}) = s_{mn}(1+\alpha_{mn}\cos(2\pi y)), \quad s_{mn}>0, \quad 0\le\alpha_{mn}<1 \\ \Phi(\xi)=\Phi(-\xi), \quad \Phi(\xi)\ge 0, \quad \int_{\Re}\Phi(\xi)d\xi = 1, \quad \Phi\in BC^1(\Re).\end{gathered} \tag{2}$$

The parameters α_{mn}, $m,n=e,i$ are referred to as the *heterogeneity parameters*. We denote the bump solutions by $U=[U_e,U_i]$. The components U_e and U_i can formally be expressed as

$$\begin{gathered}U_e(x;\alpha_e) = W_{ee}(a_e-x;\alpha_{ee}) + W_{ee}(a_e+x;\alpha_{ee}) - W_{ie}(a_i-x;\alpha_{ie}) - W_{ie}(a_i+x;\alpha_{ie}),\\ U_i(x;\alpha_i) = W_{ei}(a_e-x;\alpha_{ei}) + W_{ei}(a_e+x;\alpha_{ei}) - W_{ii}(a_i-x;\alpha_{ii}) - W_{ii}(a_i+x;\alpha_{ii}),\end{gathered}$$

where α_e and α_i are the vectors $\alpha_e=(\alpha_{ee},\alpha_{ie}])$ and $\alpha_i=(\alpha_{ei},\alpha_{ii})$ and

$$W_{mn}(\xi) = \int_0^\xi \Big(\int_0^1 \omega_{mn}(x,y)dy\Big)dx.$$

Here the *pulse width coordinates* $a_m>0$ are defined by means of the condition $U_m(\pm a_m) = \theta_m$ ($m=e,i$). Necessary conditions for the existence of the bumps read

$$f_e(a;\alpha_e)=\theta_e, \quad f_i(a;\alpha_i)=\theta_i, \tag{3}$$

where f_e and f_i are given as

$$\begin{gathered}f_e(a;\alpha_e) \equiv W_{ee}(2a_e;\alpha_{ee}) - W_{ie}(a_e+a_i;\alpha_{ie}) + W_{ie}(a_e-a_i;\alpha_{ie}),\\ f_i(a;\alpha_i) \equiv W_{ei}(a_e+a_i;\alpha_{ei}) - W_{ei}(a_i-a_e;\alpha_{ei}) - W_{ii}(2a_i;\alpha_{ii}).\end{gathered}$$

Here we have introduced the *pulse width vector* $a=(a_e,a_i)$. We obtain the following result:

Theorem 1. *Let Σ and I be the sets $\Sigma=\{(a_e,a_i);a_e,a_i>0\}$ and $I=\{(\theta_e,\theta_i); 0<\theta_m\le 1, m=e,i\}$ and $\{F_\alpha\}_{\alpha\in\mathcal{A}}$ be the 4-parameter family of vector field $F_\alpha=(f_e,f_i):\Omega\to\Re^2$ where $\alpha=(\alpha_e,\alpha_i)\in\mathcal{A}\equiv[0,1)^4$. Then the following holds true:*

1. *The set $F_\alpha(\Sigma)$ is bounded for all $\alpha\in\mathcal{A}$.*
2. *The vectorfield $F_\alpha:\Sigma\to\Re^2$ is smooth for all $\alpha\in\mathcal{A}$.*
3. *If the Jacobian $D_aF_0(a_0)$ is non-singular where a_0 is a solution of (3) when $\alpha=0$, then by the implicit function theorem the intersection between $F_\alpha(\Sigma)$ and I is non-empty i.e. there is a $k\in[0,1)$ such that*

$$F_\alpha(\Sigma)\cap I\neq\emptyset$$

for $\alpha\in\tilde{\mathcal{A}}_k$ where

$$\tilde{\mathcal{A}}_k \equiv \{\alpha\in\mathcal{A}; 0\le\alpha_{mn}<k\}\subset\mathcal{A}.$$

Based on this result, one can prove the following result:

Theorem 2. *For* $\alpha \in \tilde{\mathcal{A}}_k$ *and* $D_a F_0(a_0)$ *being non-singular, the generic picture consists of two solutions of the system (3) for each* $(\theta_e, \theta_i) \in J \equiv F_\alpha(\Sigma) \cap I$.

This result is obtained by interpreting solutions of the system (3) as a transversal intersection between two level curves and the one-to-one correspondence between the solutions of (3) and the bumps. This result means that the typical situation consists of two bumps for each $(\theta_e, \theta_i) \in J$.

We next study stability of the bumps $U_0 = (U_e, U_i)$. We write the system (1) on the compact form

$$\frac{\partial}{\partial t} U = T\left(-U + F(U)\right),$$

where F is the integral operator on the RHS of this system and

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 1/\tau \end{pmatrix}.$$

Then by imposing a perturbation on U_0 i.e. by assuming

$$U(x, y, t) = U_0(x) + V(x, y)\exp[\lambda t], \quad V = (V_e, V_i)$$

and linearizing the resulting equation for V we end up with the eigenvalue problem

$$\lambda V = G(V), \quad G(V) = T\left(-V + F'_{U_0}(V)\right).$$

Here the Frechét derivative F'_{U_0} is given by

$$F'_{U_0}(V) = \begin{pmatrix} L_{ee}V_e - L_{ie}V_i \\ \\ L_{ei}V_e - L_{ii}V_i \end{pmatrix}$$

$$(L_{mn}V_m)(x, y) = \frac{1}{|U'_m(a_m)|} \int_Y (\omega_{mn}(a_m - x, y' - y)V_m(a_m, y') + \omega_{mn}(a_m + x, y' - y)V_m(-a_m, y'))\, dy'.$$

We have the following result:

Theorem 3. *The spectrum* $Sp(G)$ *of the operator* G *can differ from* $\bigcup_{k=1,2} Sp\left(T + H^{(k)}\right)$ *only by two values,* -1 *and* $-1/\tau$. *Here the integral operator* $H^{(k)} : BC^1(Y) \times BC^1(Y) \to BC^1(Y) \times BC^1(Y)$, $k = 1, 2$ *is given as*

$$(H^{(k)}v_k)(y) = \int_Y TA^{(k)}(y' - y)v_k(y')dy', \quad v_k \in BC^1(Y) \times BC^1(Y)$$

$$A^{(1)}(y) = \begin{pmatrix} A(y) + B(y) & -C(y) - D(y) \\ \\ E(y) + F(y) & -G(y) - H(y) \end{pmatrix}$$

and

$$A^{(2)}(y) = \begin{pmatrix} A(y) - B(y) & -C(y) + D(y) \\ \\ E(y) - F(y) & -G(y) + H(y) \end{pmatrix},$$

where

$$A(y) = \tfrac{\omega_{ee}(0,y)}{|U_e'(a_e)|}, \quad B(y) = \tfrac{\omega_{ee}(2a_e,y)}{|U_e'(a_e)|}, \quad C(y) = \tfrac{\omega_{ie}(a_i-a_e,y)}{|U_i'(a_i)|}, \quad D(y) = \tfrac{\omega_{ie}(a_i+a_e,y)}{|U_i'(a_i)|},$$
$$E(y) = \tfrac{\omega_{ei}(a_e-a_i,y)}{|U_e'(a_e)|}, \quad F(y) = \tfrac{\omega_{ei}(a_e+a_i,y)}{|U_e'(a_e)|}, \quad G(y) = \tfrac{\omega_{ii}(0,y)}{|U_i'(a_i)|}, \quad H(y) = \tfrac{\omega_{ii}(2a_i,y)}{|U_i'(a_i)|}.$$

The spectrum of $H^{(k)}$, $k = 1, 2$ which is computed by means of the Fourier-decomposition method, serves as the basis for the stability method. For the scenario with two bumps for each pair of threshold values θ_e, θ_i, we find that one bump is unstable for all relative inhibition times τ and the other one is stable for small and moderate values of τ. The latter bump becomes unstable when τ exceeds a certain threshold.

R e m a r k 1. More details as well as other results can be found in Kolodina *et al.* [2].

REFERENCES

1. Nguetseng G. A general convergence result for a functional related to the theory of homogenization. *SIAM Journal on Mathematical Analysis*, 1989, vol. 20, no. 3, pp. 608-623.

2. Kolodina K., Oleynik A., Wyller J. Single bumps in a 2-population homogenized neuronal network model. *Physica D: Nonlinear Phenomena*, 2018, vol. 310, pp. 40-53.

Received 23 March 2018
Reviewed 26 April 2018
Accepted for press 5 June 2018
There is no conflict of interests.

Kolodina Karina, Norwegian University of Life Sciences, Ås, Norway, PhD-student at Faculty of Science and Technology, e-mail: karina.kolodina@nmbu.no

Oleynik Anna, University of Bergen, Bergen, Norway, PhD, Postdoctoral fellow at Department of Mathematics, e-mail: Anna.Oleynik@uib.no

Wyller John, Norwegian University of Life Sciences, Ås, Norway, PhD, Professor at Faculty of Science and Technology, e-mail: john.wyller@nmbu.no

For citation: Kolodina K., Oleynik A., Wyller J. Existence and stability of bumps in a neural field model. *Vestnik Tambovskogo universiteta. Seriya Estestvennye i tekhnicheskie nauki – Tambov University Reports. Series: Natural and Technical Sciences*, 2018, vol. 23, no. 122, pp. 131–135. DOI: 10.20310/1810-0198-2018-23-122-131-135 (In Engl., Abstr. in Russian).

DOI: 10.20310/1810-0198-2018-23-122-131-135
УДК 51-76

СУЩЕСТВОВАНИЕ И УСТОЙЧИВОСТЬ БАМПОВ В МОДЕЛИ НЕЙРОННОГО ПОЛЯ

К. Колодина[1], **А. Олейник**[2], **Й. Виллер**[1]

[1] Норвежский университет естественных наук
1433, Норвегия, Ос, ул. Университетская, 3
E-mail: karina.kolodina@nmbu.no, john.wyller@nmbu.no
[2] Бергенский университет
5020, Норвегия, Берген, ул. Аллегатен, 41
E-mail: Anna.Oleynik@uib.no

Аннотация. Исследованы существование и устойчивость *бампов* (локализованных стационарных решений) усредненной двупопуляционной модели нейронного поля в случае, когда функции активации задаются функцией типа Хевисайда.
Ключевые слова: теория усреднения; существование и устойчивость стационарных решений нелокальных моделей нейронных полей

СПИСОК ЛИТЕРАТУРЫ

1. *Nguetseng G.* A general convergence result for a functional related to the theory of homogenization // SIAM Journal on Mathematical Analysis. 1989. Vol. 20. № 3. P. 608-623.

2. *Kolodina K., Oleynik A., Wyller J.* Single bumps in a 2-population homogenized neuronal network model // Physica D: Nonlinear Phenomena. 2018. Vol. 310. P. 40-53.

Поступила в редакцию 23 марта 2018 г.
Прошла рецензирование 26 апреля 2018 г.
Принята в печать 5 июня 2018 г.
Конфликт интересов отсутствует.

Колодина Карина, Норвежский университет естественных наук, г. Ос, Норвегия, аспирант факультета науки и технологий, e-mail: karina.kolodina@nmbu.no

Олейник Анна, Бергенский университет, г. Берген, Норвегия, научный сотрудник факультета математики, e-mail: Anna.Oleynik@uib.no

Виллер Йон, Норвежский университет естественных наук, г. Ос, Норвегия, профессор факультета науки и технологий, e-mail: john.wyller@nmbu.no

Для цитирования: *Колодина К., Олейник А., Виллер Й.* Существование и устойчивость бампов в модели нейронного поля // Вестник Тамбовского университета. Серия Естественные и технические науки. Тамбов, 2018. Т. 23. № 122. С. 131–135. DOI: 10.20310/1810-0198-2018-23-122-131-135

Работа выполнена при поддержке Норвежского университета естественных наук и Научно-исследовательского совета Норвегии (проект № 239070).